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1995 J. Phys. A: Math. Gen. 28 4423

(http://iopscience.iop.org/0305-4470/28/15/021)

J. Phys. A: Math. Gen. 28 (1995) 4423–4430. Printed in the UK

Generalized linearization problems

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Received 3 February 1995

Abstract. We prove that if $P_i(x)$ and $\bar{P}_j(x)$ are two families of semi-classical orthogonal polynomials, all the linearization coefficients $L_{i,j,k}$ occuring in the product of these two families satisfy a linear recurrence relation involving only the k index. This property also extends to the linearization coefficients arising from an arbitrary number of products of semi-classical orthogonal polynomials.

1. Introduction

Let $\{P_k\}$ be a system of polynomials of degree exactly equal to k. The traditional linearization problem [1–4] consists of expanding the product P_iP_j in the $\{P_k\}$ basis $(P_r(x) \equiv P_r)$:

$$P_iP_j = \sum_{k=0}^{i+j} L_{i,j,k}P_k. \tag{1}$$

When the $\{P_k\}$ family is an orthogonal family (with respect to some positive measure $\mathrm{d}\mu(x)$), many results concerning the positivity character of the coefficients $L_{i,j,k}$ [1, 5, 6], and concerning the recurrence relation satisfied by $L_{i,j,k}$ [1, 2] are known; in some cases (classical orthogonal polynomials) the coefficients $L_{i,j,k}$ are given explicitly, very often in terms of hypergeometric functions.

In a recent paper [7], we proved that for a family of semi-classical orthogonal polynomials, the coefficients $L_{i,j,k}$ satisfy a linear recurrence relation involving only the k index, which reduces to a second-order recurrence relation for the classical family (Jacobi, Bessel, Laguerre and Hermite). More recently, Lewanowicz [8], rewriting the fourth-order differential equation for the product P_iP_j (P_i classical) given in [7], has obtained the explicit coefficient $A_i(k), i = 0, 1, 2$ of this second-order recurrence relation:

$$A_0(k)L_{i,j,k-1} + A_1(k)L_{i,j,k} + A_2(k)L_{i,j,k+1} = 0. \tag{2}$$

The aim of this work is to extend the linearization problem to two families of polynomials $P_i(x)$ and $\bar{P}_j(x)$ and to give the properties, including the recurrence relations, for the linearization coefficients (LC) defined by all the possible expansions:

$$\begin{array}{lll} P_iP_j = \sum_k L_{i,j,k}P_k & P_iP_j = \sum_k \bar{L}_{i,j,k}\bar{P}_k & P_i\bar{P}_j = \sum_k L^{\star}_{i,j,k}P_k \\ P_i\bar{P}_j = \sum_k \bar{L}^{\star}_{i,j,k}\bar{P}_k & \bar{P}_i\bar{P}_j = \sum_k L^{\star\star}_{i,j,k}P_k & \bar{P}_i\bar{P}_j = \sum_k \bar{L}^{\star\star}_{i,j,k}\bar{P}_k. \end{array} \tag{3}$$

2. Quadratic relations

Without further assumptions about the two families $P_i(x)$ and $\bar{P}_j(x)$, the LC satisfy complicated quadratic relations as do the structure constants of any algebra (even associative and commutative).

Let us write three LCs of the first two equations from the list in equation (3), obtained easily by multiplication by P_r or $\bar{P}_r$ and expanding in terms of the P_k or $\bar{P}_k$:

$$\begin{aligned}\sum_s L_{r,i,s}\sum_t L_{s,j,t} &= \sum_k L_{i,j,k}\sum_t L_{r,k,t}\\ \sum_s L_{r,i,s}\sum_t \bar{L}_{s,j,t} &= \sum_k L_{i,j,k}\sum_t \bar{L}_{r,k,t}\\ \sum_s L_{r,i,s}\sum_t L^{\star}_{s,j,t} &= \sum_k L_{i,j,k}\sum_t L^{\star}_{r,k,t}.\end{aligned} \tag{4}$$

For all LC we have the symmetry property

$$L_{a,b,c} = L_{b,a,c} \tag{5}$$

and in these quadratic relations all the summations run from 0 to $s = a + b$. Most of these relations mix the six LC taking care, however, that $\bar{L}^{\star\star}_{i,j,k}$ gives essentially the same information as $L_{i,j,k}$; $\bar{L}_{i,j,k}$ can also be compared to $L^{\star\star}_{i,j,k}$.

3. Two parameter linear relations

If we put an orthogonality structure in both P_i and $\bar{P}_j$ families, the quadratic relations for the LC become linear relations but with two mixing indices as shown below. With P_i and $\bar{P}_j$ being orthogonal families with the positive orthogonality measures $d\mu(x)$ and $d\bar{\mu}(x)$, they satisfy the recurrence relations

$$xP_i = A_iP_{i+1} + B_iP_i + C_iP_{i-1} \tag{6}$$

$$x\bar{P}_j = \bar{A}_j\bar{P}_{j+1} + \bar{B}_j\bar{P}_j + \bar{C}_j\bar{P}_{j-1}. \tag{7}$$

The multiplication of the six relations of equation (3) by x and the expansion of the results in terms of P_k and $\bar{P}_k$ give, for instance,

$$A_jL_{i,j+1,k} + B_jL_{i,j,k} + C_jL_{i,j-1,k} = A_kL_{i,j,k-1} + B_kL_{i,j,k} + C_kL_{i,j,k+1} \tag{8}$$

$$\bar{A}_jL^{\star}_{i,j+1,k} + \bar{B}_jL^{\star}_{i,j,k} + \bar{C}_jL^{\star}_{i,j-1,k} = A_kL^{\star}_{i,j,k-1} + B_kL^{\star}_{i,j,k} + C_kL^{\star}_{i,j,k+1} \tag{9}$$

$$\ldots .$$

Because the index i is fixed, we generate some kind of *cross rules* for these linear relations: the index k running on a line and the three indices j, $j+1$, $j-1$ on a column mixing three lines.

4. One parameter linear relations

A major simplification arises if we now assume that both families P_i and $\bar{P}_j$ are semi-classical. In these cases, P_i and $\bar{P}_j$ satisfy a structure relation [9]

$$\Phi(x)P_n(x)' = \sum_{k=n-s-1}^{n+t-1} C_{k,n}P_k(x) \tag{10}$$

$$\bar{\Phi}(x)\bar{P}_n(x)' = \sum_{k=n-\bar{s}-1}^{n+\bar{t}-1} \bar{C}_{k,n}\bar{P}_k(x) \tag{11}$$

where $\Phi(x)$ and $\bar{\Phi}(x)$ are polynomials of degree t and $\bar{t}$ respectively, and s and $\bar{s}$ are integers characterizing the class of families P_i and $\bar{P}_j$. $C_{k,n}$ and $\bar{C}_{k,n}$ are constants and for classical orthogonal polynomials, $s = 0$. These structure relations ensure that both families satisfy a second-order differential [10] equation written as

$$\sigma P_n''(x) + \tau P_n'(x) + \lambda_n P_n(x) = 0 \tag{12}$$

$$\bar{\sigma} \bar{P}_n''(x) + \bar{\tau} \bar{P}_n'(x) + \bar{\lambda}_n \bar{P}_n(x) = 0 \tag{13}$$

where σ, τ, λ_n (respectively $\bar{\sigma}, \bar{\tau}, \bar{\lambda}_n$) are written for $\sigma(x, n), \tau(x, n), \lambda_n(x)$.

In the classical cases, however, σ (respectively $\bar{\sigma}$) and τ (respectively $\bar{\tau}$) depend only on x and λ_n (respectively $\bar{\lambda}_n$) is independent of x and $\Phi \equiv \sigma$ (respectively $\bar{\Phi} \equiv \bar{\sigma}$).

The procedure used to generate linear recurrence relations for all LC, mixing only one index (k), follows from the two steps given below.

(i) From equations (12) and (13), there are many ways [11] to build a fourth-order linear differential equation satisfied by the product $P_i(x)\bar{P}_j(x)$. Let us call $Q_4(x, i, j)$ the corresponding differential operator:

$$Q_4(x, i, j)[y] = 0 \qquad y = P_i \bar{P}_j \tag{14}$$

$$Q_4(x, i, j) \equiv \sum_{r=0}^{4} q_r(x, i, j) D^r \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}.$$

(ii) The action of $Q_4(x, i, j)$ on P_k or $\bar{P}_k$ (cf. equation (1)) can be written as a linear, constant coefficient combination of P_k (or $\bar{P}_k$), using the following technique [7] shown for the expansion in P_k but, of course, also valid for the expansion in $\bar{P}_k$.

(a) After iteration of the the derivative, the iteration of relation (10) and (11) allows us to write

$$\Phi^r P_n^{(r)} = \sum_k D_{n,k}(r) P_k \qquad r = 2, 3, 4 \tag{15}$$

where $D_{n,k}(r)$ are constants easily computed at each step $r = 2, 3$ and 4 and we use the recurrence relation (6) or (7) for P_i. For instance, for $r = 2$ and from

$$\Phi(\Phi P_n')' = \sum_k C_{k,n} \Phi P_k' \tag{16}$$

we deduce

$$\Phi^2 P_n'' = \sum_k C_{k,n} \sum_s C_{s,k} P_s - \Phi' \sum_t C_{t,n} P_t \tag{17}$$

and we can repeat this process until $r = 4$.

(b) The multiplication of equation (14) by an appropriate integer power of Φ and $\bar{\Phi}$, say $\Psi(x) = \Phi^{\lambda}(x)\bar{\Phi}^{\bar{\lambda}}(x)$, allows us to obtain a constant coefficient linear recurrence relation in k from

$$\sum_k L_{ijk} \Psi(x) Q_4(x, i, j) P_k = 0 \tag{18}$$

and by using equation (15). The recurrence relation (6) or (7) must also be applied numerous times depending on the degree of the polynomials in front of each $\Phi^r P_k^{(r)}$. This algorithm may not give the minimal-order recurrence relation for L_{ijk} if we do not properly control the multiplicative factor $\Psi(x)$. This procedure described for L_{ijk} works for any of the generalized linearization coefficients (GLC), if the exponents λ, $\bar{\lambda}$ are chosen in an appropriate (minimal) way.

5. Classical situations

Let us now give a quick way to build the operator $Q_4(x, i, j)$ inside the classical class for P_i and $\bar{P}_j$.

From the classical differential equations [12] ($\sigma(x)$ of degree $\leqslant 2$, $\tau(x)$ of degree 1)

$$\sigma P_i'' + \tau P_i' + \lambda_i P_i = 0 \qquad \lambda_i = \frac{-i}{2}[2\tau' + (i-1)\sigma'']$$
$$\bar{\sigma}\bar{P}_j'' + \bar{\tau}\bar{P}_j' + \bar{\lambda}_j \bar{P}_j = 0 \tag{19}$$

we deduce for the product $w = P_i \bar{P}_j$ derived twice that

$$\sigma w'' + w'\tau + (\lambda_i + \bar{\lambda}_j)w = 2\sigma P_i' \bar{P}_j' + (\sigma - \bar{\sigma})\bar{P}_j'' P_i + (\tau - \bar{\tau})P_i \bar{P}_j'. \tag{20}$$

After a trivial elimination in order to keep the factors $P_i' \bar{P}_j'$ and $P_i \bar{P}_j'$ only, multiplication by $\bar{\sigma}$ allows us to obtain

$$\mathcal{D}_2[w] = A_0 P_i' \bar{P}_j' + B_0 P_i \bar{P}_j' \tag{21}$$

where

$$\mathcal{D}_2 = \bar{\sigma}\sigma D^2 + \bar{\sigma}\tau D + (\lambda_i \bar{\sigma} + \bar{\lambda}_j \sigma)I \qquad A_0 = 2\sigma\bar{\sigma} \qquad B_0 = \bar{\sigma}\tau - \bar{\tau}\sigma. \tag{22}$$

The derivative of equation (21) multiplied by $\bar{\sigma}$ will generate a third-order relation $\mathcal{D}_3[w]$, again developed in the basis $P_i' \bar{P}_j'$ and $P_i \bar{P}_j'$:

$$\mathcal{D}_3[w] = A_1 P_i' \bar{P}_j' + B_1 P_i \bar{P}_j'. \tag{23}$$

In the same way a fourth-order relation gives

$$\mathcal{D}_4[w] = A_2 P_i' \bar{P}_j' + B_2 P_i \bar{P}_j'. \tag{24}$$

$Q_4(x, i, j)$ is, therefore, given in a determinantal form as

$$Q_4(x, i, j)[w] = \begin{vmatrix} \mathcal{D}_2[w] & A_0 & B_0 \\ \mathcal{D}_3[w] & A_1 & B_1 \\ \mathcal{D}_4[w] & A_2 & B_2 \end{vmatrix} = 0. \tag{25}$$

6. Example inside classical families

As a first example, let us consider P_i classical, $\bar{P}_j = P_j'$ (also classical), and in order to simplify again we choose $i = j$:

$$w = P_i P_i' = \sum_k L_{i,i,k}^{\star} P_k = \sum_k \bar{L}_{i,i,k}^{\star} P_k'. \tag{26}$$

The equation satisfied by $P_i^2 = y$ ($P_i P_i' = \frac{1}{2}(P_i^2)'$) is of order three ($i = j$) instead of order four ($i \neq j$) and is well known [7, 8, 11]:

$$\sigma^2 y''' + 3\sigma\tau y'' + [\sigma(\tau' + 4\lambda_i) + \tau(2\tau - \sigma')]y' + 2\lambda_i(2\tau - \sigma')y = 0. \tag{27}$$

From this, coefficients $L_{i,i,k}^{\star}$ and $\bar{L}_{i,i,k}^{\star}$ can be approached in the following way.

Consider

$$P_i^2 = \sum_k \tilde{L}_{i,i,k} P_k \tag{28}$$

where $\tilde{L}_{i,i,k}$ is a solution of equation (2) with $i = j$ [7, 8]. The first derivative of equation (28) gives

$$\bar{L}_{i,i,k}^{\star} = \tfrac{1}{2}\tilde{L}_{i,i,k} \tag{29}$$

and $\bar{L}^{\star}_{i,i,k}$, therefore, satisfies equation (2). $L^{\star}_{i,i,k}$ can be found in at least two ways.

(i) From the well-known [12] representation of classical orthogonal polynomials in terms of the derivative

$$P_n = \alpha_n P'_{n+1} + \beta_n P'_n + \gamma_n P'_{n-1} \tag{30}$$

equation (26) gives

$$L^{\star}_{i,i,k} = \alpha_k \bar{L}^{\star}_{i,i,k-1} + \beta_k \bar{L}^{\star}_{i,i,k} + \gamma_k \bar{L}^{\star}_{i,i,k+1}. \tag{31}$$

(ii) The operator $Q_4(x,i,i)$ coincides *a priori* up to a polynomial factor with the operator obtained by the double derivation of equation (27) ($y' = \frac{1}{2}P_i P'_i$), giving a fourth-order equation for $w = 2y'$:

$$\sigma^2 w^{(4)} + \sigma[4\sigma' + 3\tau]w^{(3)} + \left[2\sigma'^2 + 2\sigma\sigma'' + 5\sigma'\tau + 7\sigma\tau' + 4\sigma\lambda_i + 2\tau^2\right] w''$$
$$+ w'[\sigma''\tau + 6\sigma'\tau' + 8\tau\tau' + 2\lambda_i(3\sigma' + 2\tau)] + w\tau'(4\tau' - \sigma'' + 8\lambda_i) = 0.$$

Let us recall that for $w = P_i P'_i$, the data in equation (13) are ($\bar{P}_i = P'_i$)

$$\bar{\sigma} = \sigma \qquad \bar{\tau} = \tau + \sigma' \qquad \bar{\lambda}_i = \lambda_i + \tau' \tag{32}$$

and a direct construction of equation (25) is simplified a little using the equality $\sigma = \bar{\sigma}$.

If σ is of degree two (Jacobi, Bessel), polynomials $q_r(x,i,i)$ are of degree exactly equal to r; the coefficients of D^4 are σ^2 and the coefficients of D^3 contains a factor σ. This allows us to expand $0 = Q_4(x,i,i)y = \sum_k L^{\star}_{i,i,k} Q_4(x,i,i)P_k$ in the basis of the second derivative P''_k giving, therefore, a recurrence relation in k of order five for $L^{\star}_{i,i,k}$. We proceed using the following expressions (the classical character of P_k allows us to extend all the expressions given for P_k to $P'_k \dots P^{(r)}_k$ (the recurrence relation (12), the structure relation (13), equation (32)...)):

$q_0 P_k = \sum P''_k$ (five terms)

$q_1 P'_k = \sum P''_k$ (five terms using the recurrence relation for P'_k)

$q_2 P''_k = \sum P''_k$ (five terms using the recurrence relation for P''_k)

$q_3 P'''_k = r_1\sigma(P''_k)' = \sum P''_k$ (five terms using the structure and recurrence relations for P''_k)

$q_4 P''''_k = \sigma^2(P''_k)'' = \sum P''_k$ (five terms using the structure and recurrence relations for P''_k).

It is easy to check that the expansion in terms of P_k instead of P''_k generates, in this case, a recurrence relation of order seven.

7. Regularity of Sobolev inner product

Linearization coefficients appear in a natural way in polynomials orthogonal with respect to the Sobolev inner product [13]. Let us choose a relatively general scalar product, easily extended to more than one weight:

$$(P_i, P_j) = \int_a^b \mathbf{P}_i(x)[A]\mathbf{P}_j^t(x)\rho(x)\,\mathrm{d}x \tag{33}$$

with

$$\mathbf{P}_i(x) = (P_i(x), P'_i(x), \dots, P_i^{(N)}(x))$$

and

$$[A] = [A]^t = [A_{r,s}] \qquad 0 \leqslant r, s \leqslant N \qquad A_{00} \neq 0.$$

The regularity, or the positivity, of this scalar product implies that, for all non-negative integer i,

$$(P_i, P_i) \neq 0 \qquad \text{or} \qquad (P_i, P_i) > 0.$$

These conditions on coefficients $A_{r,s}$ can be written as

$$\sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} I_i^{r,s} \neq (0) \qquad (\text{or } > 0)$$

where

$$I_i^{r,s} = \int_a^b P_i^{(r)}(x) P_i^{(s)} \rho(x)\,\mathrm{d}x \tag{34}$$

with

$$\begin{cases} a_{r,s} = a_{s,r} = \frac{1}{2} A_{r,s} & (r \neq s) \\ a_{s,s} = A_{s,s}. \end{cases}$$

Using a linearization of $P_i^{(r)}(x) P_i^{(s)}(x)$ in the form

$$P_i^{(r)} P_i^{(s)} = \sum_k L_{i,i,k}^{r,s} P_k(x) \tag{35}$$

(35) and orthogonality properties, integral (34) reduces to (monic polynomials)

$$I_i^{r,s} = L_{i,i,0}^{r,s} C_0 \qquad \text{with } C_0 = \int_a^b \rho(x)\,\mathrm{d}x. \tag{36}$$

This new linearization problem now involves three families, but is an obvious extension of the cases presented in (3) and can be written as

$$\bar{P}_i \bar{\bar{P}}_j = \sum_k l_{i,j,k} P_k \tag{37}$$

with

$$\bar{P}_i = P_i^{(r)}(x) \qquad \bar{\bar{P}}_j = P_j^{(s)}(x).$$

When $P_k(x)$ is classical and, therefore, a solution of equation (12) $\bar{P}_i$, for instance, is a solution of

$$\sigma \bar{P}_i'' + (\tau + r\sigma') \bar{P}_i' + (r - i) \left[\tau' + \frac{r + i - 1}{2} \sigma'' \right] \bar{P}_i = 0 \tag{38}$$

and $\bar{\bar{P}}_j$ a solution of the same equation with $i \to j$ and $r \to s$.

A $\bar{Q}_4$ operator annihilating the product $\bar{P}_i \bar{\bar{P}}_j$ can easily be constructed by obvious extensions of the development given in section 5. The action of $\bar{Q}_4$ on P_k as in equation (18), is particularly simple to compute using the fact that the same σ and τ are present in the three families P_k, $\bar{P}_i$ and $\bar{\bar{P}}_j$.

8. Example inside semi-classical family

As mentioned in section 4, this algorithm applies not only to the classical family but also to the very large class of semi-classical orthogonal polynomials.

However, even simple examples give very cumbersome computations. Let us comment on the following situation mixing Hermite polynomials $H_i(x)$ and generalized Hermite polynomials [14] (non-classical) $H_j^{(\eta)}(x)$, orthogonal with respect to the weight

$$|x|^{2\eta}\mathrm{e}^{-x^2} \qquad -\infty < x < +\infty \qquad (\eta \geqslant -\tfrac{1}{2}).$$

With $P_i = H_i(x)$ and $\bar{P}_j = H_j^{(\eta)}(x)$ we can consider the six cases mentioned in equation (3). Both families are symmetrical ($P_i(-x) = (-1)^i P_i(x)$, $\bar{P}_j(-x) = (-1)^j P_j(x)$), so we know already that the recurrence relation for the six linearization coefficients are of odd order (k runs on even numbers only in the even–even or odd–odd cases, and runs on odd numbers only in the even–odd or odd–even cases).

The first case $P_i P_j = \sum_k L_{i,j,k} P_k$, is the only case which belongs to the classical linearization problem and the three-term recurrence relation for $L_{i,j,k}$ is easily solved [7, 8].

In order to apply this algorithm in the five remaining cases, the data are

$$\begin{cases} \sigma = 1 \qquad \tau = -2x \qquad \lambda_n = 2n \\ \bar{\sigma} = x^2 \qquad \bar{\tau} = 2x(\eta - x^2) \qquad \bar{\lambda} = 2nx^2 - \theta_n \end{cases} \tag{39}$$

where $\theta_{2m} = 0$, $\theta_{2m+1} = 2\eta$, and in, the normalization of Chihara [14],

$$\begin{cases} A_j = \frac{1}{2} \qquad B_j = 0 \qquad C_j = n \\ \bar{A}_j = \frac{1}{2} \qquad \bar{B}_j = 0 \qquad \bar{C}_j = n + \theta_n. \end{cases} \tag{40}$$

The fourth-order operator annihilating $P_i P_j$ (second case) is already well known [15] and reads as

$$D^4 - 8xD^3 + 4\left[5x^2 - 2 + (i+j)\right]D^2 \\ +4x\left[13 - 4(i+j)\right]D + 4[(i-j)^2 + 2(i+j)(2x^2-1)]. \tag{41}$$

The operator $Q_4(x, i, j)$ annihiliating $P_i P_j$ can be constructed as in section 5 taking care now that $\bar{P}_j$ is no longer classical ($\bar{\lambda}_n$ is no longer constant for instance); in the same way $\bar{Q}_4$ annihilates $\bar{P}_i \bar{P}_j$.

Considering these five cases, the simplest cases should be the second, fifth and sixth given recurrence relations for $\bar{L}_{i,j,k}$, $L^{**}_{i,j,k}$ and $\bar{L}^{**}_{i,j,k}$.

The coefficients of all these recurrence relations are very complicated and can only be obtained using a symbolic manipulation package like MATHEMATICA.

The case $\bar{P}_i \bar{P}_j$ is treated in a survey on generalized Hermite polynomials [15]; the coefficients filled many pages!

Let us just mention that for $\bar{L}^{**}_{i,j,k}$ the recurrence relation is of order two in the even–even cases, four in the odd–odd cases, and six in the even–odd or odd–even cases.

As a final remark, let us say that the main result of this paper, which proves the existence of a linear recurrence relation, in k only, for the LC $L_{i,j,k}$ of the product of two semi-classical orthogonal polynomials P_i and $\bar{P}_j$, is obviously also true for the extended LC defined by $P_{i_1} \dots P_{i_r} = \sum_k L_{i_1,\dots,i_r,k} P_k$ where the equation satisfied by $y = P_{i_1} \dots P_{i_r}$ is of order at most $2r$.

Acknowledgments

This joint work started during a stay (December 1994) in Belgium of MNH who aknowledges the Laboratoire de Physique Mathématique of the FUNDP (Namur) for kind hospitality and financial support.

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